Geodesic metric spaces with unique blow-up almost everywhere: properties and examples

Enrico Le Donne

In this report we deal with metric spaces that at almost every point admit a tangent metric space. These spaces are in some sense generalizations of Riemannian manifolds. We will see that, at least at the level of the tangents, there is some resemblance of a differentiable structure and of (sub)Riemannian geometry. I will present some results and give examples.

Let $X = (X, d_X)$ and $Y = (Y, d_Y)$ be metric spaces. Fix $x_0 \in X$ and $y_0 \in Y$. If there exists $\lambda_j \to \infty$ such that, in the Gromov-Hausdorff convergence,

$$(X, \lambda_j d_X, x_0) \to (Y, d_Y, y_0), \qquad \text{as } j \to \infty,$$

then (Y, y_0) is called *a tangent* (or a *weak tangent*, or a *blow-up*) of X at x_0.

Some remarks are due. Fixed $x_0 \in X$, there might be more than one tangent. Moreover, in general there might not exist any tangent. However, if the distance is doubling, then, by the work of Gromov [Gro81], then tangents exists. Namely, for any sequence $\lambda_j \to \infty$, there exists a subsequence $\lambda_{j_k} \to \infty$ such that $(X, \lambda_{j_k} d_X, x_0)$ converges as $k \to \infty$. A tangent is well defined up to pointed isometry. Thus we define the set of all tangents of X at x_0 as

$$\mathrm{Tan}(X, x_0) := \{\text{tangents of } X \text{ at } x_0\}/\text{pointed isometric equivalence}.$$

We consider two questions: how big is $\mathrm{Tan}(X, x_0)$? what happens when the tangent is unique? The rough answer that we will give are the following. Under some 'standard' assumptions, if $(Y, y_0) \in \mathrm{Tan}(X, x_0)$, then $(Y, y) \in \mathrm{Tan}(X, x)$, for all $y \in Y$. Moreover, in the case of unique tangents, such tangents are very special, however, not much can be said about the initial space X.

Definition and examples. Let $(X_j, x_j), (Y, y)$ be pointed geodesic metric spaces. We write $(X_j, x_j) \to (Y, y)$ in the Gromov-Hausdorff convergence if, for all $R > 0$, we have $d_{GH}(B(x_j, R), B(y, R)) \to 0$. Here

$$d_{GH}(A, B) := \inf\{d_H^Z(A', B') : Z \text{ metric space}, A', B' \subseteq Z, A \overset{isom}{=} A', B \overset{isom}{=} B'\},$$

and $d_H^Z(\cdot, \cdot)$ is the Hausdorff distance in the space Z.

Example 1. *When $\mathbb{R}^n$ is endowed with the Euclidean distance (or more generally a norm), we have* $\mathrm{Tan}(\mathbb{R}^n, p) = \{(\mathbb{R}^n, 0)\}, \forall p \in \mathbb{R}^n$.

Example 2. *Let (M, d) be a Riemannian manifold (or more generally a Finsler manifold), we have* $\mathrm{Tan}(M, d, p) = \{(\mathbb{R}^n, \|\cdot\|, 0)\}, \forall p \in \mathbb{R}^n$.

Definition 3 (Carnot group)**.** *Let $\mathfrak{g}$ be a stratified Lie algebra, i.e., $\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$, with $[V_j, V_1] = V_{j+1}$, for $1 \le j \le s$, where $V_{s+1} = \{0\}$. Let $\mathbb{G}$ be the simply-connected Lie group whose Lie algebra is $\mathfrak{g}$. Fix $\|\cdot\|$ on V_1. Define, for any $x, y \in \mathbb{G}$,*

$$d_{CC}(x, y) := \inf\left\{\int_0^1 \|\dot\gamma(t)\| dt \mid \gamma \in C^\infty([0,1]; \mathbb{G}), \gamma(0) = x, \gamma(1) = y, \dot\gamma \in V_1\right\}.$$

The pair $(\mathbb{G}, d_{CC})$ *is called* Carnot group*.*

In particular, any Carnot group $\mathbb{G}$ is a metric space homeomorphic to the Lie group $\mathbb{G}$. Moreover, by the work of Pansu and Gromov [Pan83], the Carnot groups are the blow-downs of left-invariant Riemannnian/Finsler distances on $\mathbb{G}$. Namely, if $\|\cdot\|$ is a norm on $\mathrm{Lie}(\mathbb{G})$ extending the one on V_1 and $d_{\|\cdot\|}$ is the corresponding Finsler distance,

$$(\mathbb{G}, \lambda d_{\|\cdot\|}, 1) \overset{\lambda\to 0}{\longrightarrow} (\mathbb{G}, d_{CC}, 1).$$

Example 4. *If* $(\mathbb{G}, d_{CC})$ *is a Carnot group, then* $\mathrm{Tan}(\mathbb{G}, d_{CC}, 1) = \{(\mathbb{G}, d_{CC}, 1)\}$*. Indeed, for all* $\lambda > 0$*, there is a group homomorphism* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ *such that* $(\delta_\lambda)_*|_{V_1}$ *is the multiplication by* λ*. Consequently,* $(\delta_\lambda)_* d_{CC} = \lambda d_{CC}$*. QED*

Results. Our main theorem is the following.

Theorem 5 ([LD11])**.** *Let* (X, d) *be a geodesic metric space. Let* μ *be a doubling measure. Assume that, for* μ*-almost every* $x \in X$*, the set* $\mathrm{Tan}(X, x)$ *contains only one element. Then, for* μ*-almost every* $x \in X$*, the element in* $\mathrm{Tan}(X, x)$ *is a Carnot group.*

Example 6 (SubRiemannian manifolds)**.** *Let* M *be a Riemannian manifold (or more generally Finsler). Let* $\Delta \subseteq TM$ *be a smooth sub-bundle. Let* $\mathcal{X}^1(\Delta)$ *be the vector fields tangent to* Δ*. By induction, define* $\mathcal{X}^{k+1}(\Delta) := \mathcal{X}^k(\Delta) + [\mathcal{X}^1(\Delta), \mathcal{X}^k(\Delta)]$*. Assume that there exists* $s \in \mathbb{N}$ *such that* $\mathcal{X}^s(\Delta) = TM$ *and that, for all* k*, the function* $p \mapsto \dim \mathcal{X}^k(\Delta)(p)$ *is constant. Define, for any* $x, y \in M$*,*

$$d_{CC}(x, y) := \inf\{\mathrm{Length}(\gamma) \mid \gamma \in C^\infty([0,1]; M), \gamma(0) = x, \gamma(1) = y, \dot\gamma \in \Delta\}.$$

Then (M, d_{CC}) *is called an* (equiregular) subFinsler manifold*. In such a case, by a theorem of Mitchell, see* [Mit85, MM95]*,*

$$\mathrm{Tan}(M, d_{CC}, p) = \{(\mathbb{G}, d_{CC}, 1)\}, \qquad \forall p \in M,$$

with $(\mathbb{G}, d_{CC})$ *a Carnot group, which might depend on* p*.*

Theorem 5 is proved using the following general property.

Theorem 7 ([LD11])**.** *Let* (X, μ, d) *be a doubling-measured metric space. Then, for* μ*-almost every* $x \in X$*, if* $(Y, y) \in \mathrm{Tan}(X, x)$*, then* $(Y, y') \in \mathrm{Tan}(X, x)$*, for all* $y' \in Y$*.*

If $\#\,\mathrm{Tan}(X, x_0) = 1$, then $(Y, y_0) = (Y, y)$, for all $y \in Y$. In other words, the isometry group $\mathrm{Isom}(Y)$ acts on Y transitively. Thus we use the following.

Theorem 8 (Gleason-Montgomery-Zippin, [MZ74])**.** *Let* Y *be a metric space that is complete, proper, connected, and locally connected. Assume that the isometry group Isom*(Y) *of* Y *acts transitively on* Y*. Then Isom*(Y) *is a Lie group with finitely many connected components.*

Regarding the conclusion of the proof of Theorem 5, since moreover Y is geodesic, being X so, then Y is a subFinsler manifold, by [Ber88]. From Mitchell's Theorem and the fact that $\{Y\} = \mathrm{Tan}(Y, y)$, Y is a Carnot group. QED

Comments and more examples. There are other settings in which the tangents are (almost everywhere) unique. The snow flake metrics $(\mathbb{R}, \|\cdot\|^\alpha)$ with $\alpha \in (0,1)$ are such examples. Some examples on which the tangents are Euclidean spaces are the Reifenberg vanishing flat metric spaces, which have been considered in [CC97, DT99]. Alexandrov spaces have Euclidean tangents almost everywhere, [BGP92].

However, even in the subRiemannian setting, the tangents are not local model for the space. Indeed, there are subRiemannian manifolds with a different tangent at each point, [Var81]. In fact, there exists a nilpotent Lie group equipped with left invariant sub-Riemannian metric that is not locally biLipschitz equivalent to its tangent, see [LDOW11]. Such last fact can be seen as the local counterpart of a result by Shalom, which states that there exist two finitely generated nilpotent groups Γ and Λ that have the same blow-down space, but they are not quasi-isometric equivalent, see [Sha04].

Another pathological example from [HH00] is the following. For any $n > 1$, there exists a geodesic space X supporting a doubling measure μ such that at μ-almost all point of X the tangent is $\mathbb{R}^n$, but X has no manifold points.

References

[Ber88] Valeriĭ N. Berestovskiĭ, *Homogeneous manifolds with an intrinsic metric. I*, Sibirsk. Mat. Zh. **29** (1988), no. 6, 17–29.

[BGP92] Yuriĭ Burago, Mikhail Gromov, and Grigoriĭ Perel'man, *A. D. Aleksandrov spaces with curvatures bounded below*, Uspekhi Mat. Nauk **47** (1992), no. 2(284), 3–51, 222.

[CC97] Jeff Cheeger and Tobias H. Colding, *On the structure of spaces with Ricci curvature bounded below. I*, J. Differential Geom. **46** (1997), no. 3, 406–480.

[DT99] Guy David and Tatiana Toro, *Reifenberg flat metric spaces, snowballs, and embeddings*, Math. Ann. **315** (1999), no. 4, 641–710.

[Gro81] Mikhael Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. (1981), no. 53, 53–73.

[HH00] Bruce Hanson and Juha Heinonen, *An n-dimensional space that admits a Poincaré inequality but has no manifold points*, Proc. Amer. Math. Soc. **128** (2000), no. 11, 3379–3390.

[LD11] Enrico Le Donne, *Metric spaces with unique tangents*, accepted for publication in the Annales Academiae Scientiarum Fennicae Mathematica (2011).

[LDOW11] Enrico Le Donne, Alessandro Ottazzi, and Ben Warhurst, *Ultrarigid tangents of sub-riemannian nilpotent groups*, Preprint, submitted (2011).

[Mit85] John Mitchell, *On Carnot-Carathéodory metrics*, J. Differential Geom. **21** (1985), no. 1, 35–45.

[MM95] Gregori A. Margulis and George D. Mostow, *The differential of a quasi-conformal mapping of a Carnot-Carathéodory space*, Geom. Funct. Anal. **5** (1995), no. 2, 402–433.

[MZ74] Deane Montgomery and Leo Zippin, *Topological transformation groups*, Robert E. Krieger Publishing Co., Huntington, N.Y., 1974, Reprint of the 1955 original.

[Pan83] Pierre Pansu, *Croissance des boules et des géodésiques fermées dans les nilvariétés*, Ergodic Theory Dynam. Systems **3** (1983), no. 3, 415–445.

[Sha04] Yehuda Shalom, *Harmonic analysis, cohomology, and the large-scale geometry of amenable groups*, Acta Math. **192** (2004), no. 2, 119–185.

[Var81] A. N. Varčenko, *Obstructions to local equivalence of distributions*, Mat. Zametki **29** (1981), no. 6, 939–947, 957.